On the Internal and External View of Graded Linear Logic

Preston Keel Harley Eades III

1 Introduction

Traditionally, linear logic [4] captures the quantitative aspect of resources by forcing them to be used exactly once, but then giving a means for the programmer to control when resources can be duplicated and deleted. This is done by annotating a type, A, with a ! (read bang) as in $!A$. However, this traditional view only allows resources to be either used exactly once or any number of times. Bounded linear logic [5] was introduced to capture the spectrum of uses of resources that lie between one and any number. This is done by annotating $!A$ with a natural number, $!_nA$, which means the resource A can be used up to n times. One application of bounded linear logic has been to design type systems that only permit polynomial-time programs to type check. Bounded linear logic has since been generalzied to allow the annotation of $!A$ with an arbitrary element – called the grade – of a semiring, $r \in R$, denoted by $!_rA$ [1, 2, 3]. This generalization to Graded Linear Logic has been used to combine effects with coeffects in static type systems [2].

Throughout the literature graded linear logic appears in two forms. The first form, we call externally graded linear logic, requires that every variable be annotated with a grade. The typing judgment has the form $x_1 : A_1 \odot r_1, \ldots, x_i : A_i \odot r_i \vdash t : B$ where each $r_i \in R$ is a member of a resource algebra $(R, *, 1, +, 0, \leq)$, where addition and 0 is used to control duplication and deletion of resources. We call this form externally graded, because the grading is a property of the typing judgment, and is not part of the language. However, in the second form, called internally graded linear logic, the grading is part of the actual language. Typing judgments have the form $x_1 : A_1, \ldots, x_i : A_i \vdash t : B$, but the language has a special type form denoted by $\Box_rA$ that allows one to annotate a type with a grade $r \in R$. This type form turns out to be a comonad that respects the structure of the resource algebra.

In this talk we show that the external and internal views are two sides of the same coin, that is, one underlies the other. We do this by moving across the Curry-Howard-Lambek correspondence into the logical realm. We give a new sequent calculus that is both externally and internally graded based on Benton's Linear/Non-linear logic (LNL), where the internal modality is derived from a set of two new modalities. The new system we call Externally/Internally Graded Linear Logic consists of two fragments: one with a judgment of the form $X_1 \odot r_1, \ldots, X_i \odot r_i \vdash B$ defining the externally graded fragment, and one with a judgment of the form $(X_1 \odot r_1, \ldots, X_i \odot r_i); (A_1, \ldots, A_i) \vdash B$ defining the mixed externally/internally graded side. Then the two sides are connected via two operators F_rX which takes an element from the externally graded fragment and internalizes it into the mixed fragment, and GA which takes a linear formula and imports it into the graded fragment. Then graded modalities $\Box_rA$ can be defined in the system as $\Box_rA = F_rGA$. Thus showing that the internally graded linear logic is founded on externally graded linear logic.

Currently, graded linear logic – in both its external and internal views – are designed to precisely control how often a resource is used which in logic amounts to being able to control the structural rules for weakening (deletion) and contraction (duplication). The structural rule for exchange is often taken for granted. However, being able to control exchange has applications in software verification, functional programming, and interactive theorem proving. Thus, we ask the question, can we utilize the grading to control exchange in a similar way to weakening and contraction? We

answer this question in the positive by generalizing our sequent calculus to support the control of exchange. This new system adds a tag, $e : R \longrightarrow\!\!\!\!\!\rightarrow R$, that allows one to tag grades marking them as exchangeable. For example, $\Box_{e(1)} A$ says that A can only be used 1 time, but is exchangeable with other formulas, where $\Box_2 A$ says that A can be used two times, but cannot be exchanged with other formulas. The rules for this new sequent calculus are as follows (due to space we cannot show all the rules, but only give an idea of what the system looks like):

$$\frac{}{X \odot 1 \vdash_{\mathcal{C}} X}\,\text{id} \qquad \frac{\Phi_1, X \odot 1, \Phi_2 \vdash_{\mathcal{C}} Y}{\Phi_1, X \odot 1_{\mathsf{e}}, \Phi_2 \vdash_{\mathcal{C}} Y}\,\text{et} \qquad \frac{\Phi_1, X \odot r, Y \odot r, \Phi_2 \vdash_{\mathcal{C}} Z}{\Phi_1, (X \rhd Y) \odot r, \Phi_2 \vdash_{\mathcal{C}} Z}\,\rhd_L$$

$$\frac{\Phi_1 \vdash_{\mathcal{C}} X \quad \Phi_2 \vdash_{\mathcal{C}} Y}{\Phi_1, \Phi_2 \vdash_{\mathcal{C}} X \rhd Y}\,\rhd_R \qquad \frac{\Phi_2 \vdash_{\mathcal{C}} X \quad \Phi_1, Y \odot r_2, \Phi_3 \vdash_{\mathcal{C}} Z}{\Phi_1, (X \odot r_1 \rightharpoonup Y) \odot r_2, (r_1 * r_2) * \Phi_2, \Phi_3 \vdash_{\mathcal{C}} Z}\,\rightharpoonup_L$$

$$\frac{\Phi_2 \vdash_{\mathcal{C}} X \quad \Phi_1, Y \odot r_2, \Phi_3 \vdash_{\mathcal{C}} Z}{\Phi_1, (Y \leftharpoonup X \odot r_1) \odot r_2, (r_1 * r_2) * \Phi_2, \Phi_3 \vdash_{\mathcal{C}} Z}\,\leftharpoonup_L \qquad \frac{\Phi, X \odot r \vdash_{\mathcal{C}} Y}{\Phi \vdash_{\mathcal{C}} X \odot r \rightharpoonup Y}\,\rightharpoonup_R$$

$$\frac{X \odot r, \Phi \vdash_{\mathcal{C}} Y}{\Phi \vdash_{\mathcal{C}} Y \leftharpoonup X \odot r}\,\leftharpoonup_R \qquad \frac{\Phi \vdash_{\mathcal{L}} A}{\Phi \vdash_{\mathcal{C}} \mathsf{G}A}\,\mathsf{G}_R \qquad \frac{\Phi_1, \Phi_2 \vdash_{\mathcal{C}} Y \quad 0 \in R}{\Phi_1, X \odot 0, \Phi_2 \vdash_{\mathcal{C}} Y}\,\text{Weak}$$

$$\frac{\Phi_1, X \odot r_1, X \odot r_2, \Phi_3 \vdash_{\mathcal{C}} Y \quad (r_1 + r_2) \in R}{\Phi_1, X \odot (r_1 + r_2), \Phi_3 \vdash_{\mathcal{C}} Y}\,\text{Cont} \qquad \frac{\Phi_1, X \odot r_{1\mathsf{e}}, Y \odot r_2, \Phi_2 \vdash_{\mathcal{C}} Z}{\Phi_1, Y \odot r_2, X \odot r_{1\mathsf{e}}, \Phi_2 \vdash_{\mathcal{C}} Z}\,\text{Ex}$$

$$\frac{}{A \vdash_{\mathcal{L}} A}\,\text{id} \qquad \frac{\Gamma_1, X \odot 1, \Gamma_2 \vdash_{\mathcal{L}} A}{\Gamma_1, X \odot 1_{\mathsf{e}}, \Gamma_2 \vdash_{\mathcal{L}} A}\,\text{et} \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash_{\mathcal{L}} C}{\Gamma_1, A \rhd B, \Gamma_2 \vdash_{\mathcal{L}} C}\,\rhd_L \qquad \frac{\Gamma_1 \vdash_{\mathcal{L}} A \quad \Gamma_2 \vdash_{\mathcal{L}} B}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{L}} A \rhd B}\,\rhd_R$$

$$\frac{\Gamma_2 \vdash_{\mathcal{L}} A \quad \Gamma_1, B, \Gamma_3 \vdash_{\mathcal{L}} C}{\Gamma_1, A \rightharpoonup B, \Gamma_2, \Gamma_3 \vdash_{\mathcal{L}} C}\,\rightharpoonup_L \qquad \frac{\Gamma_2 \vdash_{\mathcal{L}} A \quad \Gamma_1, B, \Gamma_3 \vdash_{\mathcal{L}} C}{\Gamma_1, \Gamma_2, B \leftharpoonup A, \Gamma_3 \vdash_{\mathcal{L}} C}\,\leftharpoonup_L \qquad \frac{\Gamma, A \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} A \rightharpoonup B}\,\rightharpoonup_R$$

$$\frac{A, \Gamma \vdash_{\mathcal{L}} B}{\Gamma \vdash_{\mathcal{L}} B \leftharpoonup A}\,\leftharpoonup_R \qquad \frac{\Gamma_1, X \odot r, \Gamma_2 \vdash_{\mathcal{L}} A}{\Gamma_1, \mathsf{F}_r X, \Gamma_2 \vdash_{\mathcal{L}} A}\,\mathsf{F}_L \qquad \frac{\Phi \vdash_{\mathcal{C}} X}{r * \Phi \vdash_{\mathcal{L}} \mathsf{F}_r X}\,\mathsf{F}_R \qquad \frac{\Gamma_1, A, \Gamma_2 \vdash_{\mathcal{L}} B}{\Gamma_1, \mathsf{G}A \odot 1, \Gamma_2 \vdash_{\mathcal{L}} B}\,\mathsf{G}_L$$

$$\frac{\Gamma_1, \Gamma_2 \vdash_{\mathcal{L}} A \quad 0 \in R}{\Gamma_1, X \odot 0, \Gamma_2 \vdash_{\mathcal{L}} A}\,\text{Weak} \qquad \frac{\Gamma_1, X \odot r_1, X \odot r_2, \Gamma_2 \vdash_{\mathcal{L}} A \quad (r_1 + r_2) \in R}{\Gamma_1, X \odot (r_1 + r_2), \Gamma_2 \vdash_{\mathcal{L}} A}\,\text{Cont}$$

The sequent calculus is based in the Lambek Calculus [6], but then consists of an externally/internally graded linear logic.

A final generalization over existing systems is that we make the structural rules optional, but allowing for various parts of the resource algebra to be partial, or not defined at all. For example, in the Cont and Weak rules above, we require $(r_1 + r_2) \in R$ and $0 \in R$. If the former fails, then contraction is not allowed, and if the latter fails, then weakening is not allowed, but we are allowed any mixture of those. This is also true for exchange. Thus, the logic given above is a framework

for graded linear logics. We believe studying this framework will allows us to better understand substructural type systems.

References

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