On the Internal and External View of Graded Linear Logic

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All data is not created equal

- Would you pass an integer around a program freely
- What about memory pointers

Getting to Grades: Linear Logic

- Linear Logic
  - Resource conscience
  - A used exactly once, $!A$ used zero to many

$$\frac{\Gamma \vdash C}{\Gamma, !A \vdash C}\ \text{Weak} \qquad \frac{\Gamma, !A, !A \vdash C}{\Gamma, !A \vdash C}\ \text{Cont}$$

$!A$

$$\cancel{A \rightarrow A \otimes A}$$

$$!A, B \rightarrow B$$

$$A, A \rightarrow A \otimes A$$

$$!A \rightarrow A \otimes A \otimes A$$

Getting to Grades: Bounded Linear Logic

- Adding specifications for resource reuse
- Aid in time complexity

Getting to Grades: Bounded Linear Logic

- Capture between one and any number of times
- $!A \rightarrow\ !_n A$ where A can be used up to a natural number, n, times

Graded Linear Logic

- Generalize reuse bound of BLL to an arbitrary $r \in R$ of a semiring $(R, 0, +, 1, *)$ $\quad !_r A$
- Two forms: externally graded and internally graded

Graded Linear Logic

- Externally graded $x_1 : A_1 \odot r_1, \ldots, x_i : A_i \odot r_i \vdash t : B$ where $r \in R$ is of a resource algebra $(R, *, 1, +, 0, \leq)$
- Internally graded $x_1 : A_1, \ldots, A_i : x_i \vdash t : B$
    - New type annotation, $\Box_r A$

Two Sides…Same Coin

- Inference rules of linear logic parallel rules for graded linear logic

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \qquad \frac{!\Gamma \vdash B}{!\Gamma \vdash !B} \qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma, \Box_1 A \vdash B} \qquad \frac{[\Gamma] \vdash B}{r * \Gamma \vdash \Box_r B}$$

- $\Box_r A$ when eliminated introduces a discharged formula $A \odot r$ into the context

Two Sides…Same Coin

- Internally graded context contain both linear, A , and discharged, $A \odot r$, hypotheses
- Internally graded system makes use of external grade information

Two Sides…Same Coin

- New sequent calculus - Externally/Internally Graded Linear Logic
- Two fragments
  - Externally graded $X_1 \odot r_1, \ldots, X_i \odot r_i \vdash B$
  - Mixed externally/internally graded $(X_1 \odot r_1, \ldots, X_i \odot r_i); (A_1, \ldots, A_i) \vdash B$

How are they connected?

$$\frac{\Gamma_1, X \odot r, \Gamma_2 \vdash A}{\Gamma_1, F_r X, \Gamma_2 \vdash A} F_L \qquad \frac{\Gamma_1, A, \Gamma_2 \vdash B}{\Gamma_1, GA \odot 1, \Gamma_2 \vdash B} G_L$$

- $\Box_r A$ Can now be defined as $\Box_r A = F_r G A$
- Externally graded linear logic underlies internally graded linear logic

What can grades do?

$$\cancel{A_{private} \to A_{public}} \qquad A_{public} \to A_{private}$$

$$A_{[2..4]} \to A \otimes A \otimes A$$

Structural Rules

- Internally and externally graded systems precisely control resources by controlling weakening and contraction
    - Non-optional with respect to the logic
- Controlling exchange is often overlooked

Structural Rules: Exchange

- Useful when needing control over ordering
  - Mapping over lists, ordered communication
- Applications in software verification and interactive theorem proving

$\Box_{e(2)} A$ $\qquad$ $\Box_2 A$

$$\frac{\Gamma_1, X \odot e(r_1), Y \odot e(r_2), \Gamma_2 \vdash_L A}{\Gamma_1, Y \odot e(r_2), X \odot e(r_1), \Gamma_2 \vdash_L A}$$

$$\frac{\Gamma_1, X \odot e(r_1), Y \odot r_2, \Gamma_2 \vdash_L A}{\Gamma_1, Y \odot r_2, X \odot e(r_1), \Gamma_2 \vdash_L A}$$

Structural Rules: Contraction and Weakening

- Generalize system further to make these rules optional

$$\frac{\Gamma_1, \Gamma_2 \vdash A \quad 0 \in R}{\Gamma_1, X \odot 0, \Gamma_2 \vdash A} \qquad \frac{\Gamma_1, X \odot r_1, X \odot r_2, \Gamma_2 \vdash A \quad (r_1 + r_2) \in R}{\Gamma_1, X \odot (r_1 + r_2), \Gamma_2 \vdash A}$$

Controlling structural rules

- Contraction logic - no weakening
- Affine logic - no contraction
- Non-commutative and commutative versions

Where we go from here

- Understanding of substructural type systems
- Higher order systems such as dependent types
- Granule

Questions ?